

# QUANTUM MECHANICS IN THE NONCONTRACTIBLE SPACE

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**Summary.** - We show that the impact of the fundamental length in quantum mechanics can be studied within the formalism of Berry's geometrical phase with the line broadening as a resulting physical effect.

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03.65 Bz Foundations, theory of measurement, miscellaneous theories  
(including Aharonov-Bohm effect, Bell inequalities,  
Berry's phase)

03.65.Ca Formalism

In this note we explore the consequences of the hypothesis of the fundamental length in quantum mechanics motivated by the analyses of the non-contractible space in particle physics [1] and cosmology [2]. We first briefly discuss the formalism of Mead [3]. He introduced indeterminate operators in order to treat the "fundamental length algebra" in complete analogy with the Heisenberg algebra of quantum mechanics [3]:

$$(1) \quad \begin{aligned} [r_\alpha, Q_\beta] &= \imath \beta(R_c/l) \delta_{\alpha\beta}, \quad \alpha = 1, 2, 3 \\ -1 &\leq Q_\alpha \leq 1, \text{ for spectrum of } Q_\alpha, \\ \beta(y) &= \mathcal{O}(1) \text{ when } y = \mathcal{O}(1), \end{aligned}$$

$l$  = fundamental length,  $R_c$  = characteristic scale of the physical system.

For the single-particle moving in a potential field  $V(\vec{r})$ , applying "the fundamental length algebra" and the Heisenberg uncertainty principle, he found

the following uncertainty relation for Hamiltonian

$$\begin{aligned}
[r_\alpha, Q_\beta] &= \imath l \delta_{\alpha,\beta}, \\
H &= p^2/2m + V(\vec{r}), \\
(2) \quad \Delta H &\geq \frac{1}{R_c} |l \langle \vec{r} \cdot \vec{\nabla} V \rangle - (1/m) \langle \vec{p} \cdot \vec{Q} \rangle|.
\end{aligned}$$

Under the assumption that no cancellation between the two terms in the bracket takes place, one can estimate the spread of the frequency of the "broadened states" as [3]

$$(3) \quad \Delta\nu \geq \nu_0(l/R_c)\beta(R_c/l).$$

In the following we want to show that the effect of the line broadening can be correctly resolved by the inclusion of Berry's geometrical phase [4].

Following Kuratsuji and Iida [5], let us consider the Hamiltonian with a rotational symmetry consisting of the collective and internal parts:

$$(4) \quad H = H_0(P_\phi) + \begin{pmatrix} R_c & r e^{\imath\phi} \\ r e^{-\imath\phi} & -R_c \end{pmatrix},$$

and the corresponding Berry's phases for the upper and lower states are

$$(5) \quad \Gamma_\pm = \mp\pi(1 - \frac{R_c}{\sqrt{R_c^2 + r^2}}).$$

We suppose that the material point is now spread over and is represented as a patch at the pole of the sphere with a diameter equal to the fundamental length  $l$  (we assume that  $R_c \gg l$ ). The semiclassical quantization condition corrected for Berry's phase gives us the energy spectrum for the two-level system [5]:

$$\begin{aligned}
\frac{1}{2\pi} \oint_C P_\phi d\phi &= P_\phi(E) = (m - \frac{\Gamma}{2\pi})\hbar, \quad m \in \mathbb{Z}, \\
(6) \quad E_\pm^{sc} &\simeq H_0(P_\phi) \pm \sqrt{R_c^2 + r^2} \pm \frac{\hbar}{2} \frac{dH_0}{dP_\phi} \mp \frac{\hbar}{2} \frac{R_c}{\sqrt{R_c^2 + r^2}} \frac{dH_0}{dP_\phi}.
\end{aligned}$$

Our interpretation of this formula as a formula for the spectrum of the "broadened states", immediately gives us the relation for the line-broadening effect (a collection of all states with  $0 \leq r \leq l/2$  and the corresponding Berry's shifts in the spectrum define the broadened states and the broadened spectrum lines):

$$(7) \quad \Delta E \simeq \frac{\hbar}{16} \left( \frac{l}{R_c} \right)^2 \frac{dH_0}{dP_\phi}.$$

One can observe that the effect is now quadratic in  $\frac{l}{R_c}$ , contrary to the derivation of Mead with the indeterminate operators where the effect is linear in  $\frac{l}{R_c}$ . Evidently, possible cancellation in eq.(2) can take place and Mead's formalism can hardly resolve the problem, unlike the present geometrical approach that is fully compatible with quantum mechanics without any necessity to introduce a new operator algebra.

Our conclusion is valid for any quantum mechanical system if one can find a nonvanishing geometrical phase. As an example we also give the three-level system and the geometrical phase for SU(3) [6]. If two levels of this system are almost degenerate, one can evaluate the line broadening effect [6] as

$$(8) \quad \begin{aligned} \text{spectrum of internal Hamiltonian : } \mu_1 &= \frac{\sin\theta}{\sqrt{3}} + \cos\theta, \\ \mu_2 &= \frac{\sin\theta}{\sqrt{3}} - \cos\theta, \mu_3 = \frac{-2\sin\theta}{\sqrt{3}}; \\ \Gamma &= - \oint_C (\cos^2\theta d\chi_1 + \sin^2\theta \cos^2\phi d\chi_2); \\ \cos^2\theta &\simeq \left( \frac{l}{2R_c} \right)^2 \implies \Delta E \simeq \frac{\hbar}{4} \left( \frac{l}{R_c} \right)^2 \frac{dH_0}{dP_\phi}. \end{aligned}$$

We see that the pattern of the line broadening is also quadratic in  $\frac{l}{R_c}$ . Because of this fact, it could be very difficult to measure such a tiny effect even if  $R_c \sim 10^{-13} \text{cm}$  (typical scale in nuclear physics) and if  $l \sim 10^{-16} \text{cm}$ , as suggested in refs. [1] and [2].

Although the elimination of other sources of broadening could represent an insurmountable task, in the near future one can imagine some successful measurements in nuclear physics or quantum optics [3].

## References

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